



The Lee Fields Medal VI

TIME ALLOWED: UP TO TWO HOURS AND 15 MINUTES

TABLES AND CALCULATORS MAY BE USED. EACH QUESTION WORTH 10 MARKS.

1. If N has no prime factors $p \leq \sqrt{N}$, then N is prime. True or false. Justify your answer.
2. Where $0^\circ \leq \theta < 360^\circ$, show that if $z \in \mathbb{C}$ satisfies

$$z + \frac{1}{z} = 2 \cos(\theta),$$

then its modulus $|z| = 1$.

3. Solve, for $0 \leq \theta \leq 90^\circ$,

$$2^{\sin^2 \theta} + 2^{\cos^2 \theta} = 3.$$

4. Use a circle of diameter one to show that

$$\frac{4}{\sqrt{2}} < \pi < 4.$$

5. Use an isosceles triangle with side-lengths \sqrt{a} , \sqrt{a} , 2 to prove that for all $a > 1$,

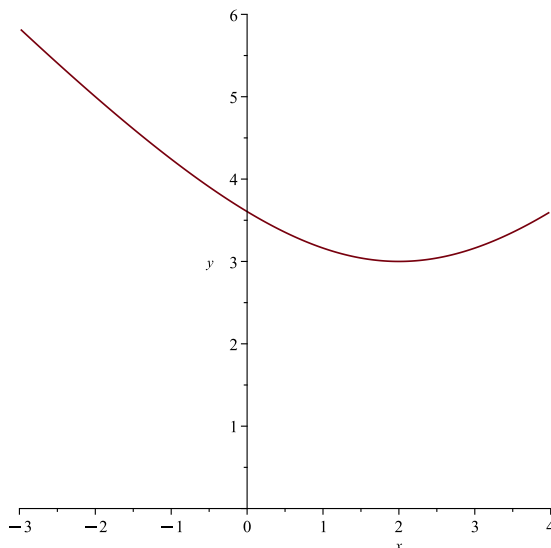
$$2 \cos^{-1} \sqrt{\frac{a-1}{a}} = \cos^{-1} \left(\frac{a-2}{a} \right).$$

6. Alice rolls one fair six-sided die. She takes the result of the roll, divides it by 6, and then takes the square root. This is her score.

Bob rolls two fair six-sided dice. He takes the higher of the two rolls and divides it by 6. This is his score.

Who do we expect to have a higher score? Justify your answer.

7. Consider the curve $y = \sqrt{x^2 - 4x + 13}$. Find the point on the curve that is closest to the origin.



8. Let $a \in \mathbb{R}$. Show that f does *not* have a local minimum at $x = 1$:

$$f(x) = x^4 - 2x^3 + ax^2 + 4x + 1.$$

9. Suppose that a professor gave a test to four students — Alice, Bob, Carol, and Dan — and wants to let them grade each other's tests. Of course, no student should grade their own test. How many ways could the professor hand the tests back to the students for grading, such that no student received their own test back?
10. An ancient Egyptian papyrus, dating from about 2000 BC and discovered only in 1853, contained the following mathematical problem. One hundred sacks of grain is to be divided between five workers in such a way that worker 2 gets more than worker 1 by the same amount as:
- worker 3 gets in excess of worker 2,
 - worker 4 gets in excess of worker 3,
 - worker 5 gets in excess of worker 4.

In addition, worker 1 and worker 2 together must get the amount that is seven times smaller than the amount the three remaining workers get together. How many sacks of grain did each worker get?